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Generalised composition law for symbolic itineraries

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Abstract. A generalised composition law is proposed which includes the $*$-composition law as a special case. The composition law preserves the order of the itineraries and their maximality. As an application the symbolic dynamics of the intermittency is briefly sketched. Other possible generalisations of the composition law are mentioned.

1. Introduction

For a unimodal map on the interval we may divide the interval into three coarse-grained parts: the centre or the critical point, the left side and the right side of the critical point. They may be denoted by the letters C, L and R, respectively. The sequence of iterates for a point x on the interval can then be associated with a sequence of the symbols L, C and R, the so-called itinerary $I(x)$ [1, 2]. The itinerary description reflects the essential feature of the evolution process, and plays a fundamental role in the construction of the MSS sequence [2].

An operation for itineraries is the $*$-product or $*$-composition law [3], which is of great importance in establishing the internal similarity and in showing the connection with renormalisation. It is our purpose here to generalise the composition law and enhance its power. In the next section we define the notation used in the paper. Section 3 is the kernel of the paper, where we propose a generalised composition law which includes the $*$-composition law as a special case. Then we show application examples of the composition law in § 4. Finally, in § 5 we make a few concluding remarks and metnion other possible generalisations for the composition law.

2. Preparation

For the unimodal map in the interval we label the two monotonic branches of the mapping function with the letters R and L, respectively. Any numerical orbit

$$x_0, x_1 = f(x_0), \ldots, x_n = f(x_{n-1}), \ldots$$

may then be associated with a symbolic sequence of the letters, in which the ith symbol is determined by where x_i is located. We may transfer the natural order of the initial points on the interval to that of their corresponding itineraries. The rule for defining the order of itineraries is then as follows. We assign an even parity to the monotonically increasing branch L, an odd parity to R, and at the same time zero parity to C for convenience. We have first the natural order

$$L < C < R.$$

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Given two itineraries $\Sigma_1 = \Sigma\mu\ldots$ and $\Sigma_2 = \Sigma\nu\ldots$, where Σ stands for the common leading string and $\mu \neq \nu$, the order of these two itineraries is then the order of μ and ν if the parity of Σ is even; otherwise it is the reverse order of μ and ν, i.e.

$$\Sigma_1 > \Sigma_2 \qquad \text{if } \mu > \nu \text{ for even } \Sigma$$

$$\Sigma_1 > \Sigma_2 \qquad \text{if } \mu < \nu \text{ for odd } \Sigma$$

It is obvious that the greater of the two strings $\Sigma\mu$ and $\Sigma\nu$ is always odd. An itinerary is termed shift maximal, or simply maximal, when any of its shifts is never greater than itself.

We now introduce the notation to be used later in this paper. For a given itinerary ΣC, where Σ consists of only R and L, we denote by $(\Sigma C)_+$ and $(\Sigma C)_-$ the greater and smaller of the two strings ΣR and ΣL. The $*$-composition law with respect to the maximal itinerary ΣC is then the substitution rule:

$$\begin{aligned} R &\to W^*(R) = \Sigma * R \equiv (\Sigma C)_+ \\ C &\to W^*(C) = \Sigma * C \equiv \Sigma C \\ L &\to W^*(L) = \Sigma * L \equiv (\Sigma C)_-. \end{aligned} \tag{2.1}$$

Under this transformation rule a given maximal itinerary $\Delta C = \delta_0\delta_1\ldots\delta_n C$ is mapped to its 'coarse-grained' image

$$W^*(\Delta C) \equiv \Sigma * \Delta C = W^*(\delta_0)W^*(\delta_1)\ldots W^*(\delta_n)W^*(C). \tag{2.2}$$

The remarkable properties of the $*$-composition law have been discussed in [3].

Finally, we shall denote by $|\Sigma|$ the length of the string Σ, by $(\Sigma)_n$ with $n < |\Sigma|$ the string consisting of the first n symbols of Σ, and by $\Sigma|_C$ the finite string Σ with its last symbol replaced by the letter C.

3. A generalised composition law

The $*$-composition law preserves the parity of strings and the maximality of itineraries. This is the hint to generalise the construction. Instead of the substitution rule (2.1) we can consider a more general one

$$\begin{aligned} R &\to W(R) = \rho = r_1 r_2 \ldots r_u \\ L &\to W(L) = \lambda = l_1 l_2 \ldots l_v \end{aligned} \tag{3.1}$$

where the finite strings ρ and λ consisting of only R and L satisfy the following conditions:

(i) ρ is odd, and λ is even
(ii) $\rho > \lambda$
(iii) $\rho|_C$ is maximal
(iv) $\rho\lambda|_C$ is maximal
(v) $\rho\lambda^\infty$ is maximal.

According to the period window theorem [4] (that the maximality of $\Sigma|_C$ implies maximality of $(\Sigma R)^\infty$ and $(\Sigma L)^\infty$) and its modified reverse theorem, a superstable itinerary can be deduced from its adjacent non-superstable ones, so we need not consider the rule for the letter C in (3.1). It is easy to see that the $*$-composition law satisfies all the five conditions, and that from the conditions (i) and (ii) the composition law preserves the order of itineraries. If we denote the transformation of the itinerary $\Sigma = s_0 s_1 s_2 \ldots$ by

$$\Sigma \to W(\Sigma) = W(s_0) W(s_1) W(s_2) \ldots \tag{3.2}$$

then $\Sigma < \Delta$ implies $W(\Sigma) < W(\Delta)$.

We now come to our main proposition. The transformation preserves the maximality of itineraries, i.e. the transformed itinerary of a maximal itinerary other than L^∞ is still maximal. The itinerary L^∞ should be excluded since λ^∞ need not be maximal. The maximality of ρ^∞ is obvious from condition (iii) and the period window theorem, so for Σ we shall consider only itineraries starting with RL.

Before proving the proposition let us deduce from the conditions that $\rho\lambda^m|_c$ is maximal for any integer m. We first show the maximality of $\rho\lambda^2|_c$. From the maximality of $\rho\lambda|_c$ it is obvious that, for $k \geqslant |\rho\lambda| = u + v$

$$\rho\lambda^2|_c > \mathcal{S}^k(\rho\lambda^2|_c). \tag{3.3}$$

If $k < |\rho\lambda|$ then from the maximality of $\rho\lambda|_c$ the relation (3.3) is true for $(\rho\lambda)_{u+v-k} \neq \mathcal{S}^k(\rho\lambda)$ where $(\rho\lambda)_{u+v-k}$ indicates the meaning of the first $u+v-k$ symbols of $\rho\lambda$. When $(\rho\lambda)_{u+v-k} = \mathcal{S}^k(\rho\lambda) \equiv \alpha$ the string α must be odd from the maximality of $\rho\lambda|_c$. Next we compare $\mathcal{S}^{u+v-k}(\rho\lambda^\infty)$ and λ^∞, the strings after the common leading string α. From the maximality of $\rho\lambda^\infty$ the relation (3.3) is valid whenever $\beta \equiv (\mathcal{S}^{u+v-k}(\rho\lambda^\infty))_v \neq \lambda$. However, when $\beta = \lambda$, the string $\alpha\beta = \alpha\lambda$ must be odd because α is odd and λ is even, hence we have the validity of (3.3). Thus, $\rho\lambda^2|_c$ is maximal. Regarding $\rho\lambda$ as a new ρ, we can similarly prove the maximality of $\rho\lambda^3|_c$, and hence by induction that of any $\rho\lambda^m|_c$.

It is necessary to make a few ramarks about the conditions.

(i) The maximality of $\rho|_c$ and $\rho\lambda|_c$ does not imply the maximality of $\rho\lambda^2|_c$; for example $\rho = RL$ and $\lambda = LRLRLL$.

(ii) The maximality of $\rho\lambda|_c$ does not imply the maximality of $\rho|_c$; for example $\rho = RLRRRLR$ and $\lambda = LRR$.

(iii) The maximality of $\rho|_c$ and $\rho\lambda^\infty$ does not imply the maximality of $\rho\lambda|_c$; for example $\rho = RLRR$ and $\lambda = LRLR$.

We now prove the proposition. We see first that from the maximality of Σ and conditions (i) and (ii) for $k_0 = \Sigma_{i=0}^{n-1} |W(s_i)|$, we have directly

$$W(\Sigma) \geqslant \mathcal{S}^{k_0}(W(\Sigma)). \tag{3.4}$$

We only need to consider $k = k_0 + j$ where $0 < j < |W(s_n)|$. We consider different situations separately as follows.

Case I. $\mathcal{S}^n(\Sigma) = s_n s_{n+1} \ldots = RLR \ldots$

From the period window theorem [4] the maximality of $\rho\lambda|_c$ implies the maximality of $(\rho\lambda)^\infty$. Thus, when $\rho\lambda \neq \alpha\beta$ where $\alpha = \mathcal{S}^k(\rho\lambda)$ and $\beta = (\rho)_k$ we have always

$$W(\Sigma) > \mathcal{S}^k(W(\Sigma)). \tag{3.5}$$

It is impossible that $\rho\lambda = \alpha\beta$, otherwise from the maximality of $\rho\lambda|_c$ we would deduce that both α and β are odd, which contradicts the fact that $\alpha\beta = \rho\lambda$ is odd.

Case II. $s_n s_{n+1} \ldots = RL^m R \ldots$

From the maximality of Σ the string Σ has to start with RL^m, i.e. $\Sigma = RL^m \ldots$. The proof is then similar to that for case I with λ^m substituted for λ.

Case III. $s_n s_{n+1} \ldots = LR \ldots$

From the maximality of $(\rho\lambda)^\infty$ when $\alpha \equiv (\rho\lambda)_{v-k} \neq \mathcal{S}^k(\lambda) \equiv \alpha'$ relation (3.5) is valid. If $\alpha = \alpha'$ the maximality of $\rho\lambda|_c$ then implies that α is odd. The maximality of $\rho\lambda^2|_c$ then implies $\rho\lambda^2 \ldots \geqslant \alpha\lambda \ldots > \alpha\rho \ldots$ because α is odd and $\rho > \lambda$. Thus, (3.5) is always true.

Case IV. $s_n s_{n+1} \ldots = L^m R \ldots$

The proof for this case is similar to that for case III with λ^m substituted for λ.

Case V. $s_n s_{n+1} \ldots = R^m \ldots$

From the maximality of ρ^∞ expression (3.5) is true for $\rho \neq \alpha\beta$ where $\alpha = \mathcal{S}^k(\rho)$ and $\beta = (\rho)_{u-k}$. If $\rho = \alpha\beta$, from the maximality of $\rho|_c$ we would have an odd α and at the same time an odd β. Thus, $\rho = \alpha\beta$ would become even, which contradicts the condition that ρ is odd, so relation (3.5) is verified.

We have considered all the possible cases, so the proof of the proposition is completed.

4. Examples of applications

We beiefly show a few application examples of the generalised composition law. We do not attempt a very general discussion, and in the following examples adopt the negative Schwarzian derivative assumption to reduce possible map 'pathologies'.

4.1. A working definition for coarse-grained chaos

The largest kneading sequence or the itinerary $I(f(C))$ is RL^∞. The chaotic behaviour of the dynamics at the parameter corresponding to RL^∞ is fairly clear. In this case the map is topologically conjugated to the shift map of the sequence space on the two symbols [5], and almost any periodic or non-periodic itinerary corresponds to a point on the interval. If for a given map there exists a point on the interval whose itinerary can be written in the form $\rho\lambda^\infty$ with the finite strings ρ and λ satisfying the above five conditions required for the generalised composition law, then a correspondence can be built between any itinerary consisting of R and L (belonging to the map of RL^∞) and its coarse-grained itinerary consisting of ρ and λ (belonging to the given map). The symbols R and L in the former itinerary are replaced by the finite strings ρ and λ, respectively. In this way the latter is generated from the former. Taking RL^∞ to be a prototype of chaos, we can call $\rho\lambda^\infty$ a coarse-grained chaos.

For the unimodal map with negative Schwarzian derivatives the periodic attractor, if it exists, must attract the critical point [1]. When the kneading sequence is of the form $\rho\lambda^\infty$ which cannot be reduced to a periodic sequence, i.e. here λ^∞ is an unstable periodic point, then there exists no stable periodic orbit. The critical point C can be attracted by no trivial attractor, and after a finite number of iterates it collides with the unstable periodic point λ^∞. The critical point is said to be homoclinic to the unstable periodic point, and the so-called crisis then occurs. We can have a working definition for coarse-grained chaos by examining the kneading sequence. That is, the kneading sequence of the coarse-grained chaos is of the form $\rho\lambda^\infty$ or ρQ, where Q is an unstable quasiperiodic point which may be viewed as the limit of a series of unstable periodic points. The stange attractor now appears. It consists of the above-mentioned coarse-grained orbits, and is physically visible, i.e. is of a finite measure.

The Feigenbaum itinerary $R^{*\infty}$ is a quasiperiodic chain of the 'atoms' R and L obtained by continuing the transformation

$$\mathscr{T}: R \to RL \quad L \to RR \tag{4.1}$$

for an infinite time, i.e.

$$R^{*\infty} = \lim_{n\to\infty} \mathscr{T}^n(R). \tag{4.2}$$

At the same time we also have

$$R^{*\infty} = \lim_{n\to\infty} \mathscr{T}^n(RL^\infty). \tag{4.3}$$

When we say that the Feigenbaum kneading sequence is the onset of chaos, it is in a sense of the coarse-grained chaos. That is, a $\rho\lambda^\infty$-type kneading sequence first appears immediately above the Feigenbaum kneading sequence.

4.2. *The strange repeller at period three*

The kneading sequence of period three first appears to be $(RLR)^\infty$. We can write $(RLR)^\infty = R(LRR)^\infty \equiv \rho\lambda^\infty$ with $\rho = R$ and $\lambda = LRR$. It is easy to check that ρ and λ satisfy the five conditions for the generalised composition law. Thus, any itinerary of the symbols R and L at the map of RL^∞ has an image itinerary of the symbols ρ and λ. In this way we obtain a coarse-grained chaotic set which, however, forming a strange repeller, is now of zero measure [1, 5].

There is another way to write $(RLR)^\infty$ in the form $\rho\lambda^\infty$, i.e. $(RLR)^\infty = RL(RRL)^\infty$, which corresponds to the choice of $\rho = RL$ and $\lambda = RRL$. In this case the conditions for the generalised composition law are also satisfied.

4.3. *Period three implies chaos revisited*

It is well known that if there exists a period three orbit, stable or unstable, then there exists a periodic orbit for any period n. In terms of the symbolic dynamics we can explicitly 'name' these orbits. When a period three point exists, there is certainly a point on the interval whose itinerary is $(RLR)^\infty$. We have written $(RLR)^\infty = R(LRR)^\infty = RL(RRL)^\infty$. According to the generalised composition law we can construct coarse-grained itineraries from itineraries at the kneading sequence RL^∞. Thus, we have for $n = 0, 1, 2, \ldots$

$$\begin{aligned} &(RL^n)^\infty \to [R(LRR)^n]^\infty && \text{with period } 3n+1 \\ &(RL^n)^\infty \to [RL(LRR)^n]^\infty && \text{with period } 3n+2 \end{aligned} \tag{4.4}$$

and

$$(RL^{n+1}RR)^\infty \to [R(LRR)^{n+1}RR]^\infty \qquad \text{with period } 3(n+1). \tag{4.5}$$

We have obtained periodic orbits for all possible lengths (although odd periodic itineraries also correspond to period-double orbits).

4.4. *Calculation of toplogical entropy for period three*

By relating a symmetric tent map to a sawtooth map and further to a shift map Milnor and Thurston have proposed an algorithm to obtain the topological entropy from the kneading determinant [6]. Here we show a direct way of calculating the entropy from the fundamental strings for the kneading sequences. As we know, the knowledge of the number N_n of periodic orbits of order n allows an estimate of the topological entropy K [7], i.e.

$$K = \lim_{n\to\infty} \left(\frac{1}{n} \ln N_n\right). \tag{4.6}$$

From the two ways of writing $(RLR)^\infty$ in the form $\rho\lambda^\infty$, i.e. $(RLR)^\infty = R(LRR)^\infty = RL(RRL)^\infty$, one can see for the map of $(RLR)^\infty$ that all itineraries may be written in terms of only two fundamental strings R and RL. The number of periodic orbits of order n is then the same as the number of ways for a boy to climb up n steps if every time he may jump up either one or two steps. It is easy to write out the recursion relation for N_n:

$$N_{n+1} = N_n + N_{n-1}. \tag{4.7}$$

This is the same as the recursion relation for the Fibonacci sequence. We have

$$N_n \sim \omega^n \tag{4.8}$$

where $\omega = \frac{1}{2}(\sqrt{5}+1)$ is the golden mean. Thus, $K = \ln \omega = 0.481\,211\ldots.$

Similarly, for the kneading sequence $(RL^mR)^\infty$ the fundamental strings are

$$R, RL, RL^2, \ldots, RL^m.$$

The recursion relation is then

$$N_{n+m+1} = N_{n-m} + N_{n+m-1} + \ldots + N_n \tag{4.9}$$

from which the topological entropy can be obtained.

4.5. The symbolic dynamics of intermittency

Let us examine the intermittency before period three. There is an increasing series of periodic kneading sequences $[R(LRR)^m]^\infty$, $m = 1, 2, \ldots$, which approaches $(RLR)^\infty$ for $m \to \infty$. There is another series of kneading sequences $R[(LRR)^mRR]^\infty$ which are of the $\rho\lambda^\infty$ type and increasing in m, and which also approaches $(RLR)^\infty$ for $m \to \infty$. For those non-periodic $\rho\lambda^\infty$-type kneading sequences the coarse-grained chaos can be seen to be of non-zero measure. This is the intermittency. Up to now the intermittency has been viewed as a transient phenomenon. The symbolic dynamics provides a new point of view and a more precise description. There is a sandwich structure of periodic windows and coarse-grained chaos. The power m is a natural measure of the laminar length, which is not a clearly defined quantity in the previous theory for the intermittency.

5. Conclusion

In the above we have proposed a generalised composition law which preserves the order of itineraries and their maximality, and includes the $*$-composition law as a special case. By means of this composition law a working definition of coarse-grained chaos is introduced, and the chaotic set of zero measure, the strange repeller at period three, is described. Furthermore, we have shown how to 'name' a periodic orbit of any given order n in the presence of period three, and how to calculate the topological entropy directly from symbols. Among many other possible applications we have mentioned that the intermittency can be better described in this new language.

There are other possibilities of generalisation for the composition law. These new composition laws might be useful in analysing the nesting structure of the bifurcation diagram. For example, instead of the sequence

$$R^{\infty} < (RL)^{\infty} < (RL^2)^{\infty} < \ldots < RL^{\infty}$$

between the itineraries $RL(RR)^{\infty} = RLR(RR)^{\infty}$ and $(RLR)^{\infty}$ there is a 'reverse' sequence

$$\rho^{\infty} > (\rho\lambda)^{\infty} < (\rho\lambda^2)^{\infty} > \ldots > \rho\lambda^{\infty}$$

where $\rho = RLR > \lambda = RR$, and both ρ and λ are even. Such reverse sequences can be found in many places of the bifurcation diagram. We have also constructed the Fibonacci sequence of itineraries. Let us state the 'rule of the game': $\Phi^{(0)} = L$, $\Phi^{(1)} = R$. If $\rho = \max(\Phi^{(m)}, \Phi^{(m-1)})$ and $\lambda = \min(\Phi^{(m)}, \Phi^{(m-1)})$, then $\Phi^{(m+1)} = \rho\lambda$. We claim that $\Phi^{(m)}|_c$, $m = 0, 1, 2, \ldots$, is maximal. The applications of these generalised composition laws to the bahaviour of dyanmical systems is under study.

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